Étale theta functions, mono-theta environments, and [IUTchI] §1-§3, I

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We start from : an initial Θ-datum

An **initial Θ-datum** (initial Θ-data in the original paper) is a 7-tuple

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon})$$

satisfying the following 7 conditions.
Condition (1). F : a number field s.t. $-1 \in F$, and $\overline{F}$: an alg. closure of F.

Condition (2)

$X_F = E_F \setminus \{0\}$, E_F: ell. curve $/F$ s.t.

- F is Galois over $F_{\mathrm{mod}} := \mathbb{Q}(j(E_F))$.
- E_F has good or semistable red. at any $v \nmid \infty$.
- $E_F[6](\overline{F}) = E_F[6](F)$.
  ($\Longrightarrow E_F$ has good or split semistable red. at any $v \nmid \infty$.)

Condition (3)

ℓ: a prime number ≥ 5, s.t. the image of

$$\rho_{E_F,\ell} : G_F := \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{\mathbb{F}_\ell}(E[\ell](\overline{F})) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$$

contains $\mathrm{SL}_2(\mathbb{F}_\ell)$. Set

$$\mathrm{Ker}\,\rho_{E_F,\ell} =: \mathrm{Gal}(\overline{F}/K).$$

($\Longrightarrow K/F$: Galois). We moreover assume:

- $C_K := (X_F/\pm 1) \otimes_F K$ is a K-core, i.e. any étale morphism over K between finite étale coverings of C_K is over C_K.

Condition (4)

$\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$: a finite set of places of F_{mod} s.t.

- $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \neq \emptyset$
- $v \in \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \Longrightarrow v \nmid 2\ell\infty$
- $v \in \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \Longrightarrow j(E_F) \notin \mathcal{O}_{F_{\mathrm{mod},v}}$
  ($\Longrightarrow X_F$: bad red. at any $w|v$.)

We do not assume that $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$ contains all the bad primes.

Condition (5)

$\underline{C}_K = \underline{X}_K/\{\pm 1\}$, where
$\underline{X}_K \to X_K = X_F \otimes_F K$: cyclic cov. of deg. ℓ, unramified at the boundary.

$-1 : \underline{X}_K \xrightarrow{\cong} \underline{X}_K$: a lift of $-1 : X_K \xrightarrow{\cong} X_K$
(a lift $-1 \longleftrightarrow$ a cusp $\underline{0}$ of $\underline{X}_K$).

So $(\underline{X}_K, \underline{0})$ comes from an isogeny $\underline{E}_K \to E_K$ of elliptic curves.
$(\underline{X}_K, \underline{0})$: unique up to isom. since $\mathrm{Image}\,\rho_{E_F,\ell}$: large.

$\rightsquigarrow \underline{\underline{X}}_{\overline{F}}, \underline{\underline{C}}_{\overline{F}}$ over $\overline{F}$.

$\underline{\underline{\Delta}}_X \twoheadrightarrow \underline{\underline{\Delta}}_X^{\Theta} \twoheadrightarrow \overline{\Delta}_X, \sharp\overline{\Delta}_X = \ell^3.$

$$1 \to \overline{\Delta}_{\Theta} \to \overline{\Delta}_X \to \overline{\Delta}_X^{\mathrm{ell}} \to 1,$$

$\overline{\Delta}_{\Theta} \cong \mathbb{Z}/\ell\mathbb{Z}, \overline{\Delta}_X^{\mathrm{ell}} \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2}.$

$\underline{X}_{\overline{F}} \to X_{\overline{F}}$: degree $\ell \longleftrightarrow \overline{\Delta}_{\underline{X}} \subset \overline{\Delta}_X$: index ℓ.

$$1 \to \overline{\Delta}_{\Theta} \to \overline{\Delta}_{\underline{X}} \to \overline{\Delta}_{\underline{X}}^{\mathrm{ell}} \to 1,$$

$\overline{\Delta}_{\underline{X}}^{\mathrm{ell}} \subset \overline{\Delta}_X^{\mathrm{ell}}, \overline{\Delta}_X^{\mathrm{ell}}/\overline{\Delta}_{\underline{X}}^{\mathrm{ell}} \cong \mathbb{Z}/\ell\mathbb{Z}.$

Recall: Construction of $\underline{\underline{X}}_{\overline{F}}$ (continued)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{\Delta}_{\Theta} & \longrightarrow & \overline{\Delta}_{X} & \longrightarrow & \overline{\Delta}_{X}^{\mathrm{ell}} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \overline{\Delta}_{\Theta} & \longrightarrow & \overline{\Delta}_{\underline{X}} & \longrightarrow & \overline{\Delta}_{\underline{X}}^{\mathrm{ell}} & \longrightarrow & 1.
\end{array}$$

$\overline{\Delta}_{\underline{X}}$: abelian, $\{\pm 1\} \circlearrowleft \overline{\Delta}_{\underline{X}}$
$\rightsquigarrow \overline{\Delta}_{\underline{X}} = \overline{\Delta}_{\Theta} \times \overline{\Delta}_{\underline{X}}^{\mathrm{ell}}$.

$\underline{\underline{X}}_{\overline{F}} \to \underline{X}_{\overline{F}}$: degree $\ell \longleftrightarrow \overline{\Delta}_{\underline{\underline{X}}} = \overline{\Delta}_{\underline{X}}^{\mathrm{ell}} \subset \overline{\Delta}_{\Theta}$ index ℓ.

Condition (6) : $\underline{\epsilon}$: a non-zero cusp of $\underline{C}_K$

$\rightsquigarrow$ $\underset{\rightarrow}{X}_K$, $\underset{\rightarrow}{C}_K$ as follows:

$^{\exists!}f$: rational function on $\underline{C}_K$ whose values at $0, \underline{\epsilon}$, "$[2](\underline{\epsilon})$" are $\infty, 0, 1$, respectively and which has a simple zero at $\underline{\epsilon}$.

$\underset{\rightarrow}{X}_K$, $\underset{\rightarrow}{C}_K$: the (orbi)curve obtained from $\underline{X}_K$, $\underline{C}_K$ by adjoining a ℓ-th root of f.

Condition (7)

$\underline{\mathbb{V}}$: a set of places of K s.t.

- The composite

$$\begin{aligned}\underline{\mathbb{V}} &\hookrightarrow (\text{the places of } K)\\ &\to (\text{the places of } F_{\mathrm{mod}}) =: \mathbb{V}_{\mathrm{mod}}\end{aligned}$$

  is bijective

- Set

$$\underline{\mathbb{V}}^{\mathrm{bad}} := \{\underline{v} \in \underline{\mathbb{V}} \mid \underline{v}|^{\exists} v \in \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}\}.$$

  Then for any $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, $\underline{C}_v = \underline{C}_K \otimes_K K_v$: type $(1, \mathbb{Z}/\ell\mathbb{Z})_{\pm}$.

An aim

For an initial Θ-datum

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon})$$

and for $\underline{v} \in \underline{\mathbb{V}}$, we will construct

$$\underline{\underline{\mathcal{F}}}_{\underline{v}}, \mathcal{C}_{\underline{v}}, \mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}, \mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}}$$

where ...

An aim (continued)

where

- $\underline{\underline{\mathcal{F}}}_{\underline{v}}$: a Frobenioid when $\underline{v} \nmid \infty$
- $\mathcal{C}_{\underline{v}}$, $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$: $p_{\underline{v}}$-adic (resp. archimedean) Frobenioids if $\underline{v} \nmid \infty$ (resp. $\underline{v} | \infty$) (so its divisor monoid is monoprime).
- $\tau^{\vdash}_{\underline{v}}$, $\tau^{\Theta}_{\underline{v}}$: characteristic splittings ($\doteqdot$ splitting of the inclusion of functors "$\mathcal{O}^{\times} \subset \mathcal{O}^{\triangleright}$") of $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$

On Next Slides

Construction of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is hard when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. In the next slides we will focus on this case.

Bad local situation

K : a finite ext. of $\mathbb{Q}_p$.
E : a Tate elliptic curve / K s.t.

- $E[2](\overline{K}) = E[2](K)$
- $X^{\log} = (X, 0)$: not K-arithmetic

$\mathfrak{X}^{\log}$: stable model of $X^{\log}$ over $\mathfrak{S}^{\log} = (\mathrm{Spec}\,\mathcal{O}_K, \text{the closed point})$.
(so the special fiber of $\mathfrak{X}^{\log}$ is irreducible with one singular point, at which the formal completion is of the form $\mathcal{O}_K[[u,v]]/(uv - q_E)$)
$Y^{\log} \to X^{\log}$: the topological $\underline{\mathbb{Z}}$-covering

Fix an odd integer $\ell \geq 5$ satisfying
$E[\ell](\overline{K}) = E[\ell](K)$.
$Y^{\log} \to \underline{X}^{\log} \to X^{\log}$, $\deg(\underline{X}) = \ell$.
$\rightsquigarrow \underline{\underline{X}}^{\log}, \underline{\underline{Y}}^{\log}$.
We have a cartesian diagram:

$$\begin{array}{ccccc}
\underline{\underline{Y}}^{\log} & \longrightarrow & \underline{\underline{X}}^{\log} & & \\
\downarrow & & \downarrow & & \\
Y^{\log} & \longrightarrow & \underline{X}^{\log} & \longrightarrow & X^{\log}.
\end{array}$$

Choose $\dot{X}^{\log} \to X$, degree 2, unramified at 0. Taking composite with $\dot{X}$, we have

$$\begin{array}{ccccc} \underline{\underline{\ddot{Y}}}^{\log} & \longrightarrow & \underline{\underline{\dot{X}}}^{\log} & & \\ \downarrow & & \downarrow & & \\ \ddot{Y}^{\log} & \longrightarrow & \underline{\dot{X}}^{\log} & \longrightarrow & \dot{X}^{\log}. \end{array}$$

Action of ± 1

We can take a lift of $-1 : X \to X$ to an involution of

$$\begin{tikzcd}
\underline{\underline{\dot{X}}}^{\log} \arrow[r] \arrow[d] & \underline{\dot{X}}^{\log} \arrow[r] \arrow[d] & \dot{X}^{\log} \arrow[d] \\
\underline{\underline{X}}^{\log} \arrow[r] & \underline{X}^{\log} \arrow[r] & X^{\log}
\end{tikzcd}$$

Passing to the quotient by $\{\pm 1\}$ we have

$$\begin{tikzcd}
\underline{\underline{\dot{C}}}^{\log} \arrow[r] \arrow[d] & \underline{\dot{C}}^{\log} \arrow[r] \arrow[d] & \dot{C}^{\log} \arrow[d] \\
\underline{\underline{C}}^{\log} \arrow[r] & \underline{C}^{\log} \arrow[r] & C^{\log}.
\end{tikzcd}$$

The Next Slides

In the bad local situation, we will construct a Frobenioid

$$\underline{\underline{\mathcal{F}}}.$$

Recall: a **Frobenioid** is a category with some additional structures. (In our case, it turns out that the additional structures can be recovered from the underlying category. We often regard $\underline{\underline{\mathcal{F}}}$ just as a category.)

Frobenioid

Recall: a Frobenioid is a quadruple $(\mathcal{F}, \mathcal{D}, \Phi, \mathcal{F} \to \mathbb{F}_\Phi)$, where

- $\mathcal{F}$: a category,
- $\mathcal{D}$: a connected, totally epimorphic cat. ($=:$ E-cat.)
- Φ : a divisorial monoid on $\mathcal{D}$ ($\rightsquigarrow \mathbb{F}_\Phi$ the associated category)
- $\mathcal{F} \to \mathbb{F}_\Phi$: a covariant functor,

that satisfies a lot of technical conditions. The underlying category is $\mathcal{F}$. The category $\mathcal{D}$ is called the **base category**, and Φ the **divisor monoid**.

A Typical Example

$\mathcal{D}$: a cat. of connected regular noeth. schemes.
Assume $\mathcal{D}$: E-cat.
Φ: the monoid on $\mathcal{D}$ given by

$$\Phi(X) = (\text{effective divisors on } X).$$

$\rightsquigarrow$ a Frobenioid $\mathcal{F}$ defined as follows:

A Typical Example (continued)

$\mathcal{F}$: category of pairs $(X, \mathcal{L})$ where

- X : an object of $\mathcal{D}$
- $\mathcal{L}$ an invertible $\mathcal{O}_X$-module

A morphism $\phi : (X, \mathcal{L}) \to (Y, \mathcal{L}')$ is a triple $(\phi_{\mathcal{D}}, n_\phi, \iota_\phi)$ where

- $\phi_{\mathcal{D}} : X \to Y$: a morphism of $\mathcal{D}$
- n_ϕ : an integer ≥ 1
- $\iota_\phi : \mathcal{L}^{\otimes n_\phi} \hookrightarrow \phi_{\mathcal{D}}^* \mathcal{L}'$ an injection

This gives an example of model Frobenioids.

Model Frobenioids

To a quadruple $(\mathcal{D}, \Phi, \mathbb{B}, \mathrm{Div}_{\mathbb{B}})$ where

- $\mathcal{D}$: E-category
- Φ : a divisorial monoid on $\mathcal{D}$ ($\rightsquigarrow \Phi^{\mathrm{gp}}$ is also a monoid on $\mathcal{D}$)
- $\mathbb{B}$: a group-like monoid on $\mathcal{D}$
- $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \Phi^{\mathrm{gp}}$: homomorphism,

we can associate the following Frobenioid $\mathcal{F}$.

Model Frobenioids (continued)

... we can associate the following Frobenioid $\mathcal{F}$.

Objects: pairs (A, α) of $A \in \mathrm{Obj}(\mathcal{D})$, $\alpha \in \Phi^{\mathrm{gp}}(A)$

Morphisms: a morphism $(A, \alpha) \to (B, \beta)$ is a quadruple $(\phi_{\mathcal{D}}, Z_\phi, n_\phi, u_\phi)$, where $(\phi_{\mathcal{D}}, Z_\phi, n_\phi)$ is a morphism of $\mathbb{F}_\Phi$ and $u_\phi \in \mathbb{B}(A)$, such that

$$n_\phi \alpha + Z_\phi = \phi_{\mathcal{D}}^* \beta + \mathrm{Div}_{\mathbb{B}}(u_\phi)$$

in $\Phi(A)^{\mathrm{gp}}$.

Such a Frobenioid $\mathcal{F}$ is called a model Frobenioid.
The Frobenioid $\underline{\underline{\mathcal{F}}}$ that we would like to construct in the bad local situation is a model Frobenioid.

Construction of $\underline{\underline{\mathcal{F}}}$ (1) : the base category

We construct the base category $\mathcal{D}$.

Let $\mathcal{D} = \mathbb{B}^{\mathrm{tp}}(\underline{\underline{X}})^{\circ}$: cat. of connected tempered coverings of $\underline{\underline{X}}^{\log}$.

Construction of $\underline{\underline{\mathcal{F}}}$ (2) : the divisor monoid

We construct the divisor monoid Φ on $\mathcal{D}$.

For any object B of $\mathcal{D}$, let

B^{ell} : the maximal subcovering of the composite $B \to \underline{\underline{X}}^{\log} \to X^{\log}$ s.t. B^{ell} is unramified at the cusp of $X^{\log}$.

Construction of $\underline{\underline{\mathcal{F}}}$ (2) (continued)

The divisor monoid is roughly

$$B \mapsto \Phi(B) := \text{``}\mathrm{DIV}_+(\mathfrak{B}^{\mathrm{ell}})^{\mathrm{pf}}\text{''},$$

where

$\mathfrak{B}^{\mathrm{ell}}$: the stable model of B^{ell}.

DIV_+ : the effective divisors supported on the union of the special fiber and the cusps

pf : perfection (e.g., $(\mathbb{Z}_{\geq 0})^{\mathrm{pf}} = \mathbb{Q}_{\geq 0}$)

N.B. The actual definition is more complicated.

Construction of $\underline{\underline{\mathcal{F}}}$ (2) (remark)

We regard $\Phi(B)$ as a submonoid of Φ_0, defined roughly as

$$\Phi_0(B) = \text{“}\mathrm{DIV}_+(\mathfrak{B})^{\mathrm{pf}}\text{”}.$$

when B admits a suitable stable model $\mathfrak{B}$.

N.B. The actual definition is more complicated, and is given by introducing the notion of divisors on universal combinatorial coverings and then by doing some “sheafification” process.

Construction of $\underline{\underline{\mathcal{F}}}$ (3) : the remaining structure

We construct the group-like monoid $\mathbb{B}$ on $\mathcal{D}$.

$$\mathbb{B}_0 : B \mapsto \text{“}\{f : \text{log-merom. on } \mathfrak{B}\}\text{”}.$$

when B admits a suitable stable model $\mathfrak{B}$.
N.B. The actual definition is more complicated,
Here log-merom. = mero. func. f on B s.t. for ${}^{\forall}N$,
f admits a N-th root in a tempered covering of B.

$$\mathbb{B}(B) = \{f \in \mathbb{B}_0(B) \mid \mathrm{div}(f) \in \Phi(B)\}.$$

$\mathrm{Div}_{\mathbb{B}}$: the restriction of div.

Properties of $\underline{\underline{\mathcal{F}}}$

$(\mathcal{D}, \Phi, \mathbb{B}, \mathrm{Div}_{\mathbb{B}}) \rightsquigarrow$ a model Frobenioid $\underline{\underline{\mathcal{F}}}$.

The Frobenioid $\underline{\underline{\mathcal{F}}}$ has the following properties:

- $\mathcal{D}$: slim, of FSMFF type
- Φ : perfect, perf-factorial, non-dilating, cuspidally pure, rational
- $\underline{\underline{\mathcal{F}}}$: of unit-profinite type, of isotropic type, of model type, of sub-quasi-Frobenius trivial type, not of group-like type, of standard type, of rationally standard type

(I will not explain the terminology appeared here.)

Consequence

As a consequence, we can reconstruct

$$\mathcal{D}, \Phi, \text{ and } \underline{\underline{\mathcal{F}}} \to \mathbb{F}_{\Phi} \to \mathcal{D}$$

category theoretically from $\underline{\underline{\mathcal{F}}}$.

The Base-Field-Theoretic Hull $\mathcal{C}$

$\mathbb{F}_0 \subset \mathbb{B}_0$; submonoid of constant functions.
$\Longrightarrow \mathbb{F}_0 \subset \mathbb{B} \subset \mathbb{B}_0$.

$$\Phi^{\mathrm{bs-fld}} := \mathbb{Q}_{>0} \cdot \mathrm{Image}(\mathbb{F}_0 \to \Phi_0^{\mathrm{gp}}) \cap \Phi.$$

$(\mathcal{D}, \Phi^{\mathrm{bs-fld}}, \mathbb{F}_0, \mathbb{F}_0 \to (\Phi^{\mathrm{bs-fld}})^{\mathrm{gp}})$
$\rightsquigarrow$ the model Frobenioid $\mathcal{C}$ (called the **base-field-theoretic hull** of $\underline{\underline{\mathcal{F}}}$).

One can reconstruct $\mathcal{C}$ category theoretically from $\underline{\underline{\mathcal{F}}}$.

Next Slides

We will go back to the global situation.

In later pages, we will give a closer look at the bad local situation.

Let us go back to our first setting, i.e.

An initial Θ-datum

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon})$$

is given.

For every $\underline{v} \in \mathbb{V}$, we will construct $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ (which is a Frobenioid when $\underline{v} \nmid \infty$), $p_{\underline{v}}$-adic or archimedean Frobenioids $\mathcal{C}_{\underline{v}}$, $\mathcal{C}_{\underline{v}}^{\vdash}$, $\mathcal{C}_{\underline{v}}^{\Theta}$ and characteristic splittings $\tau_{\underline{v}}^{\vdash}$, $\tau_{\underline{v}}^{\Theta}$ of $\mathcal{C}_{\underline{v}}^{\vdash}$ and $\mathcal{C}_{\underline{v}}^{\Theta}$.

We divide the situation into the following three cases

- $\underline{v} \in \mathbb{V}^{\mathrm{bad}}$
- $\underline{v} \notin \mathbb{V}^{\mathrm{bad}}$, $\underline{v} \nmid \infty$
- $\underline{v} | \infty$

Construction of the Frobenioids (1) : when $\underline{v} \in \mathbb{V}^{\mathrm{bad}}$

$\underline{\underline{\mathcal{F}}}_{\underline{v}}$: the Frobenioid $\underline{\underline{\mathcal{F}}}$ in the bad local situation. In particular its base category is $\mathcal{D}_{\underline{v}} = B^{\mathrm{tp}}(\underline{\underline{X}}_{\underline{v}})^{\circ}$.
$\mathcal{C}_{\underline{v}} \subset \underline{\underline{\mathcal{F}}}$: the base-field theoretic hull.
Set $\mathcal{D}^{\vdash}_{\underline{v}} := B(\mathrm{Spec}\, K_{\underline{v}})^{\circ}$.

We have an adjunction

$$\mathcal{D}_{\underline{v}} \leftrightarrows \mathcal{D}^{\vdash}_{\underline{v}}$$

Next Slides

We will construct a Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ whose base category is $\mathcal{D}^{\vdash}_{\underline{v}}$. We need preliminaries.

Recall: $\mathcal{O}^{\triangleright}(\ )$ and $\mathcal{O}^{\times}(\ )$

For a Frobenioid $\mathcal{F}$ and an object A of $\mathcal{C}$, we set

$\mathcal{O}^{\triangleright}(A) := \{f : A \to A \mid f \mapsto (\mathrm{id}, *, 1)$ under $\mathcal{F} \to \mathbb{F}_{\Phi}\}$,

$\mathcal{O}^{\times}(A) = (\mathcal{O}^{\triangleright}(A))^{\times}$, and
$\mu_N(A) = \mathrm{Ker}(N : \mathcal{O}^{\times}(A) \to \mathcal{O}^{\times}(A))$.

Notation

For $W \in \mathrm{Obj}(\mathcal{D}_{\underline{v}})$, let $\mathbb{T}_W = (W, 0)$ denote the Frobenius trivial object lying over W.

We use the subscript $\underline{v}$ to denote objects of $\mathrm{Obj}(\mathcal{D}_{\underline{v}})$ introduced in the bad local situation, when $X^{\log} = (E_{K_{\underline{v}}}, 0)$.

$\underline{\underline{q}}_{\underline{v}} := q_{\underline{v}}^{1/2\ell}$

$q_{\underline{v}}$: the q-parameter of $E_{K_{\underline{v}}}/K_{\underline{v}}$. We regard $q_{\underline{v}}$ as an element of $\mathcal{O}^{\triangleright}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$

The assumption on $E_F[2]$ and the definition of K $\Longrightarrow q_{\underline{v}}$ admits a 2ℓ-th root $\underline{\underline{q}}_{\underline{v}} := q_{\underline{v}}^{1/2\ell}$ in $\mathcal{O}^{\triangleright}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$.

$\mathcal{C}^{\vdash}_{\underline{v}}$ and $\tau^{\vdash}_{\underline{v}}$

$\underline{\underline{q}}_{\underline{v}}$ defines the constant section $\mathbb{N}_{\mathcal{D}_{\underline{v}}} \hookrightarrow \mathcal{C}_{\underline{v}}$ of $\Phi_{\mathcal{C}_{\underline{v}}}$: the divisorial monoid for $\mathcal{C}_{\underline{v}}$. Denote this section by $\log(\underline{\underline{q}}_{\underline{v}})$.

Set

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} = \mathbb{N} \cdot \log(\underline{\underline{q}}_{\underline{v}})|_{\mathcal{D}^{\vdash}_{\underline{v}}} \subset \Phi_{\mathcal{C}_{\underline{v}}}|_{\mathcal{D}^{\vdash}_{\underline{v}}}.$$

$\rightsquigarrow$ ($p_{\underline{v}}$-adic) $\mathcal{C}^{\vdash}_{\underline{v}}$ whose base category is $\mathcal{D}^{\vdash}_{\underline{v}}$.

$\underline{\underline{q}}_{\underline{v}} \in K_{\underline{v}} \rightsquigarrow \underline{\underline{q}}_{\underline{v}}$ defines a characteristic splitting $\tau^{\vdash}_{\underline{v}}$ modulo $\mu_{2\ell}$.

Next Slides

We will construct a Frobenioid $\mathcal{C}_{\underline{v}}^{\Theta}$ and its base category is $\mathcal{D}_{\underline{v}}^{\Theta}$. We again need preliminaries.

$\underline{\underline{\Theta}}_{\underline{v}}$

Let us regard $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ as an object of $\mathcal{D}_{\underline{v}}$ via $\underline{\underline{\ddot{Y}}}_{\underline{v}} \to \underline{\underline{X}}_{\underline{v}}$.

$\underline{\underline{\Theta}}_{\underline{v}} \in \mathcal{O}^{\times}(\mathbb{T}^{\div}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$: the inverse of the Frobenioid theoretic ℓ-th root of theta function. Here the superscript $\div$ means the biratioalization (i.e., localization with respect to the pre-steps).

Remark. Relation of $\underline{\underline{\Theta}}_{\underline{v}}$ with $\underline{\underline{q}}$

We have $\underline{\underline{\Theta}}_{\underline{v}}(\sqrt{-q_{\underline{v}}}) = \underline{\underline{q}}_{\underline{v}}$.
(Note. Both $\underline{\underline{\Theta}}_{\underline{v}}$ and $\underline{\underline{q}}_{\underline{v}}$ are determined only up to $\mu_{2\ell}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$.)

The base category $\mathcal{D}^{\Theta}_{\underline{v}}$

$\mathcal{D}^{\Theta}_{\underline{v}} \subset (\mathcal{D}_{\underline{v}})_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$ the full subcat. whose obj. are the products of objects of $\mathcal{D}^{\vdash}_{\underline{v}}$ and $\underline{\underline{\ddot{Y}}}_{\underline{v}}$.

$\Longrightarrow \mathcal{D}^{\vdash}_{\underline{v}} \cong \mathcal{D}^{\Theta}_{\underline{v}}$: equivalence.

Define $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}$: monoid on $\mathcal{D}^{\Theta}_{\underline{v}}$ as

$$A^{\Theta} \mapsto \mathcal{O}^{\times}(\mathbb{T}_{A^{\Theta}})(\underline{\underline{\Theta}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}})^{\mathbb{N}} \subset \mathcal{O}^{\times}(\mathbb{T}^{\div}_{A^{\Theta}}).$$

$$\Longrightarrow \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \cong \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) := \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)^{\times}.$$

The Frobenioid $\mathcal{C}^{\Theta}_{\underline{v}}$

$$\Longrightarrow \mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \cong \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-),\ \underline{\underline{q}}_{\underline{v}} \mapsto \underline{\underline{\Theta}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}}.$$

$\rightsquigarrow$ p-adic Frobenioid $\mathcal{C}^{\Theta}_{\underline{v}} \subset \underline{\underline{\mathcal{F}}}^{\div}_{\underline{v}}$ whose base cat. is $\mathcal{D}^{\Theta}_{\underline{v}}$ and a characteristic splitting $\tau^{\Theta}_{\underline{v}}$ modulo $\mu_{2\ell}$ such that

$$\mathcal{F}^{\vdash}_{\underline{v}} := (\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}) \cong \mathcal{F}^{\Theta}_{\underline{v}} := (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}}).$$

Theorem

We can reconstruct the followings category theoretically from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$:

- $\mathcal{D}_{\underline{v}}$, $\mathcal{D}^{\vdash}_{\underline{v}}$, $\mathcal{D}^{\Theta}_{\underline{v}}$
- $\mathcal{C}_{\underline{v}}$, $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$
- $\tau^{\vdash}_{\underline{v}}$, $\tau^{\Theta}_{\underline{v}}$